Drawing Planar Graphs via Dessins d'Enfants

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Belyi Maps

Definition

A rational function $\beta(z) = \frac{p(z)}{q(z)}$, where $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, is a Belyi Map if it has at most **three critical values**, say $\{\omega^{(0)}, \omega^{(1)}, \omega^{(\infty)}\}$.

Remarks:

- $\mathbb{P}^1(\mathbb{C})$ refers to the complex projective line, that is the set $\mathbb{C} \cup \{\infty\}$.
- An $\omega \in \mathbb{P}^1(\mathbb{C})$ is a critical value of $\beta(z)$ if $\beta(z) = \omega$ for some $\beta'(z) = 0$.

Belyi Maps

Question

Is $\beta(z) = 4z^5(1 - z^5)$ a Belyi Map?

1. Form a polynomial equation

$$\beta(z) = \frac{\omega_1}{\omega_0} \qquad \Longleftrightarrow \qquad \omega_0\, 4z^5(1 - z^5) - \omega_1 = 0$$

2. Compute the discriminant

$$\text{disc}\ \left[\omega_0\, 4z^5(1 - z^5) - \omega_1\right] = (\mathit{constant})\ \omega_1^4\, \omega_0^9\, (\omega_1 - \omega_0)^5$$

3. Find the roots

$$= \left\{ \frac{\omega_1}{\omega_0} \in \mathbb{P}^1(\mathbb{C}) \,\middle|\, \omega_1^4\, \omega_0^9\, (\omega_1 - \omega_0)^5 = 0 \right\} = \{0, 1, \infty\}$$

Answer

Yes, $\beta(z)$ is a Belyi Map with critical values $\{0, 1, \infty\}$

Dessin d'Enfant

Definition

Given a Belyi map $\beta(z) = p(z)/q(z)$ we consider the preimages

$$\begin{aligned} B &= \text{"black" vertices} && = \beta^{-1}(0) \\ W &= \text{"white" vertices} && = \beta^{-1}(1) \\ E &= \text{edges} && = \beta^{-1}([0,1]) \\ F &= \text{midpoints of faces} && = \beta^{-1}(\infty) \end{aligned}$$

We define the bipartite graph $\Delta_\beta = (B \cup W, E)$ as the **Dessin d'Enfant**.

Remarks:

- A bipartite graph is a collection of vertices and edges where the vertices are placed into two disjoint sets, none of whose elements are adjacent
- Following Grothendieck, "Dessin d'Enfants" is French for "Children's Drawings"

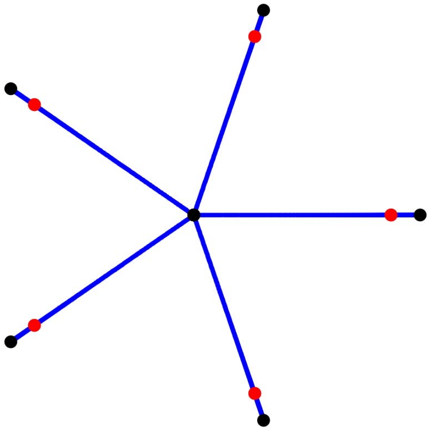
Dessin d'Enfant

Example

$$\beta(z) = 4z^5(1 - z^5)$$

- We found that β is a Belyi map with critical values $0, 1, \infty$.
- We consider its preimages

$$\begin{aligned} B &= \beta^{-1}(0) \\ &= \{0\} \cup \{5^{th} \text{ roots of } 1\} \\ W &= \beta^{-1}(1) \\ &= \{5^{th} \text{ roots of } 1/2\} \\ F &= \beta^{-1}(\infty) \\ &= \{\infty\} \end{aligned}$$



Research Question

Motivating Question

Let Γ be a connected planar graph. Can we find a Belyi map $\beta(z)$ such that Γ is the Dessin d'Enfant of this map?

Remarks:

- A graph is connected if there is a path between any two points
- A graph is planar if it can be drawn without any edges crossing
- Any planar graph can be seen as a bipartite graph if we label all vertices as "black" and label all midpoints of edges as "white"

Specific Question

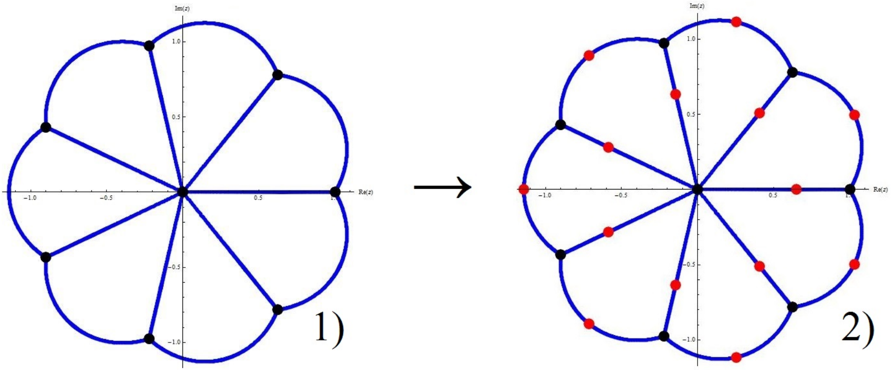
Given a specific web or a tree, can we explicitly find its corresponding Belyi map?

Methodology

From a graph to a bipartite graph:

1) Label graph's vertices as "black"

2) Add "white" vertices as midpoints of edges

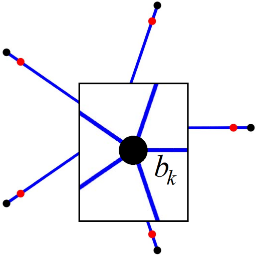


From a bipartite graph to a Belyi map:

3) Label "black" vertices as

$$B = \{b_1, b_2, ..., b_r\} \subseteq \mathbb{P}^1(\mathbb{C})$$

such that each point b_k has e_k edges incident.



Since we want $B = \beta^{-1}(0)$, we must have

$$p(z) = (constant) \prod_{k=1}^{r} (z - b_k)^{e_k} \quad \text{where} \quad \beta(z) = \frac{p(z)}{q(z)}$$

Methodology

4) Label "white" vertices as

$$W = \{w_1, w_2, ..., w_n\} \subseteq \mathbb{P}^1(\mathbb{C})$$

such that each point w_k has $\underline{2}$ edges incident.
Since we want $W = \beta^{-1}(1)$, we must have

$$p(z) - q(z) = (constant) \prod_{k=1}^{n} (z - w_k)^2 \quad \text{where} \quad \beta(z) = \frac{p(z)}{q(z)}$$

5) Label midpoints of faces vertices as

$$F = \{f_1, f_2, ..., f_s\} \subseteq \mathbb{P}^1(\mathbb{C})$$

such that each point f_k has d_k edges that enclose it.
Since we want $F = \beta^{-1}(\infty)$, we must have

$$q(z) = (constant) \prod_{k=1}^{s} (z - f_k)^{d_k}$$

Methodology

Proposition

If there exist constants $b_k,\ w_k,\ f_k,\ p_0,\ q_0,\ r_0 \in \mathbb{C}$ such that

$$\underbrace{p_0 \prod_{k=1}^{r} (z - b_k)^{e_k}}_{\text{vertices}} - \underbrace{q_0 \prod_{k=1}^{n} (z - w_k)^2}_{\text{edges}} - \underbrace{r_0 \prod_{k=1}^{s} (z - f_k)^{d_k}}_{\text{faces}} = 0$$

for all z, then the rational function

$$\beta(z) = -\frac{p_0}{r_0} \frac{\prod_{k=1}^{r} (z - b_k)^{e_k}}{\prod_{k=1}^{s} (z - f_k)^{d_k}}$$

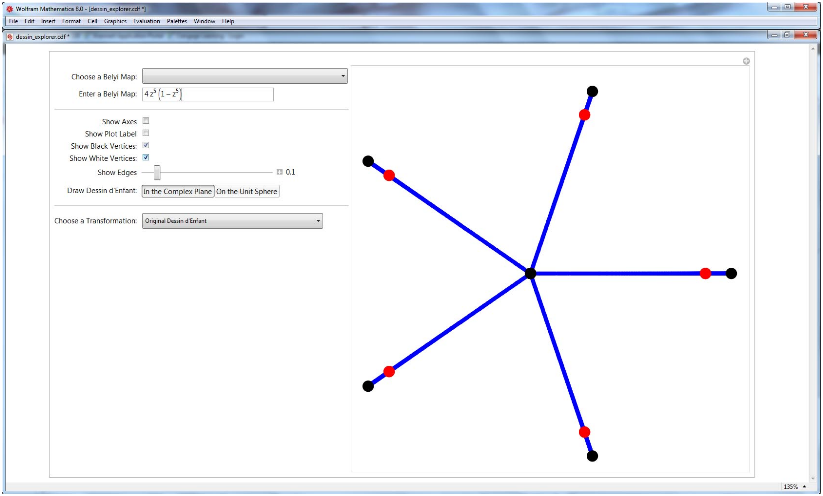
is a Belyi map of degree

$$2\,n = \sum_{k=1}^{r} e_k = \sum_{k=1}^{s} d_k .$$

List of Results

Webs	Cycles
	Dipoles
	Prisms
	Bipyramids
	Antiprisms
	Trapezohedrons
	Wheels
	Gyroelongated Bipyramid
	Truncated Trapezohedron
Trees	Paths
	Stars

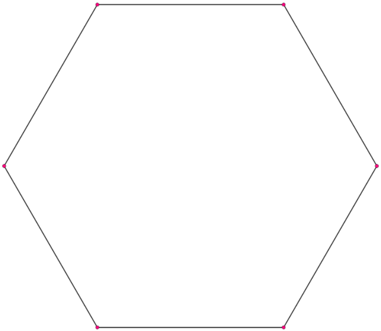
Graphing: Mathematica Notebook



http://www.math.purdue.edu/∼egoins/site//Dessins%20d'Enfants.html

Cycles

$$\beta(z) = -\frac{(z^n - 1)^2}{4z^n}$$



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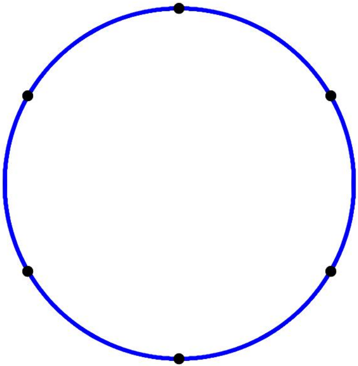
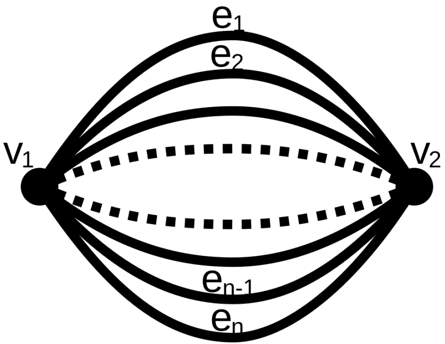


Mathematica Notebook

Dipoles

$$\beta(z) = -\frac{4z^n}{(z^n - 1)^2}$$



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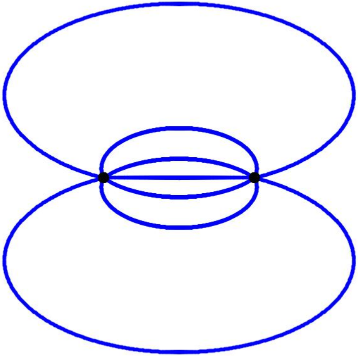
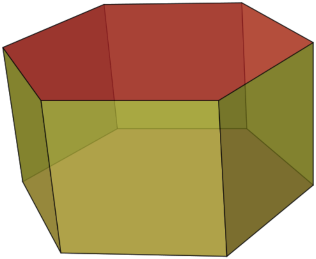


Mathematica Notebook

Prisms

$$\beta(z) = \frac{(z^{2n} + 14z^n + 1)^3}{108\, z^n (z^n - 1)^4}$$



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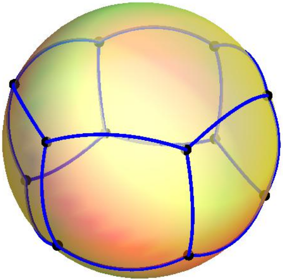
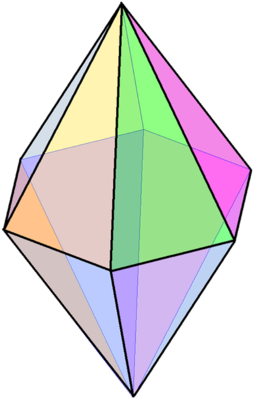


Mathematica Notebook

Bipyramids

$$\beta(z) = \frac{108z^n(z^n - 1)^4}{(z^{2n} + 14z^n + 1)^3}$$



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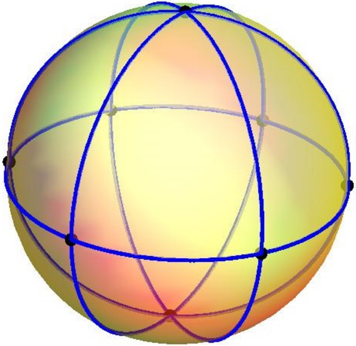
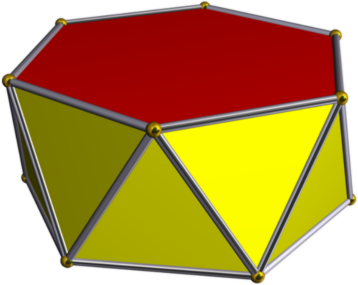


Mathematica Notebook

Antiprisms

$$\beta(z) = \frac{(z^{2n} + 10z^n - 2)^4}{16(z^n - 1)^3(2z^n + 1)^3}$$



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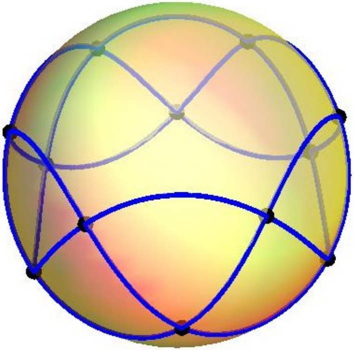
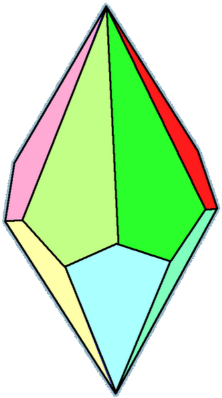


Mathematica Notebook

Trapezohedron

$$\beta(z) = \frac{16(z^n - 1)^3(2z^n + 1)^3}{(z^{2n} + 10z^n - 2)^4}$$



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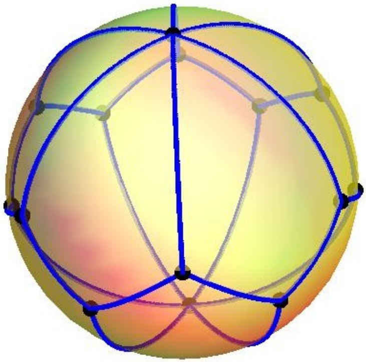
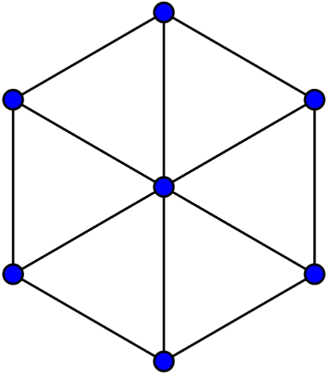


Mathematica Notebook

Wheels

$$\beta(z) = -\frac{64z^n(z^n-1)^3}{(8z^n+1)^3}$$



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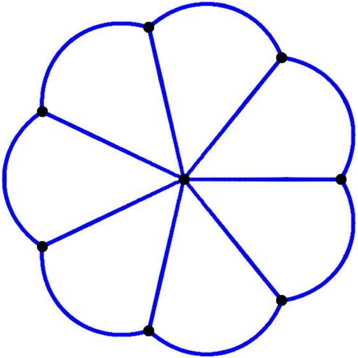
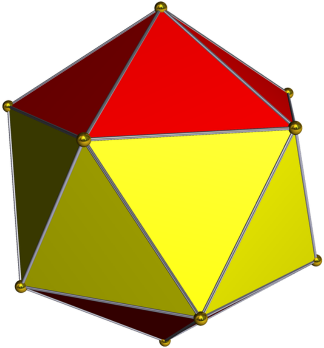


Mathematica Notebook

Gyroelongated Bipyramid

$$\beta(z) = -\frac{1728z^n(z^{2n}+11z^n-1)^5}{(z^{4n}-228z^{3n}+494z^{2n}+228z^n+1)^3}$$



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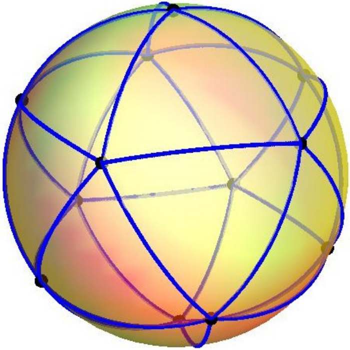
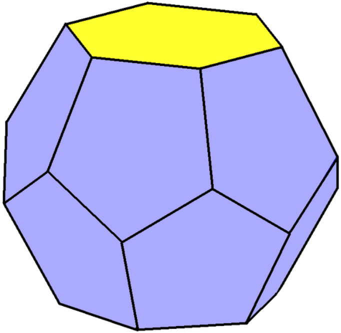


Mathematica Notebook

Truncated Trapezohedron

$$\beta(z) = -\frac{(z^{4n} - 228z^{3n} + 494z^{2n} + 228z^{n} + 1)^3}{1728z^n(z^{2n} + 11z^n - 1)^5}$$



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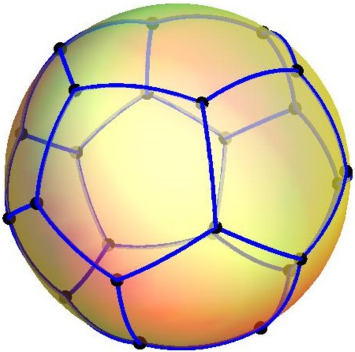


Mathematica Notebook

Paths

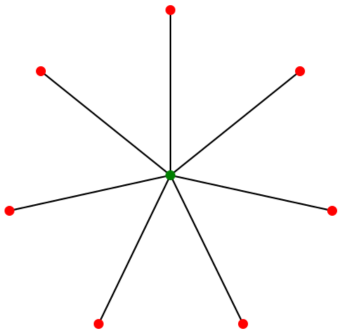
$$\beta(z) = \sin^2(n \cos^{-1} z)$$

Wikipedia

Mathematica Notebook

Stars

$$\beta(z) = 4z^n(1 - z^n)$$



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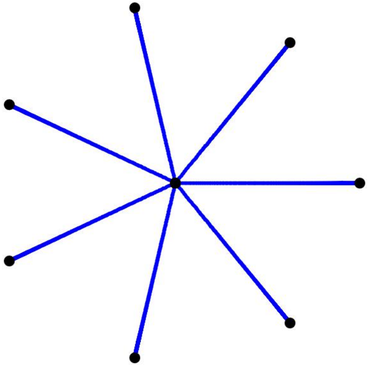


Mathematica Notebook

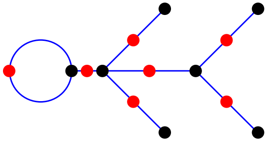
Further Research

- Find the Belyi maps for the following:

Elongated Pyramid	Gyroelongated Pyramid
Rotunda	Elongated Bipyramid
Truncated Bipyramid	Bicupola
Birotunda	

- Create a `Mathematica` notebook which will generate Belyi maps for any given tree or web.

- Find a Belyi map for the Stick Figure



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